

Probing Quantum General Relativity through Exactly Soluble Midi-Superspaces II: Polarized Gowdy Models

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Abstract

Canonical quantization of the polarized Gowdy midi-superspace with a 3-torus spatial topology is carried out. As in an earlier work on the Einstein-Rosen cylindrical waves, symmetry reduction is used to cast the original problem in 4-dimensional space-times to a 3-dimensional setting. To our knowledge, this is the first complete, systematic treatment of the Gowdy model in the geometrodynamical setting.

1 Introduction

In the canonical approach, general relativity is described by a suitable set of phase space variables subject to constraints. The topology of the space-time manifold is fixed once and for all to be $\Sigma \times \mathbb{R}$, and the basic phase space variables are fields on Σ . Examples of such settings are provided by geometrodynamics, where the phase space variables are the spatial metric of Σ and its conjugate momentum (essentially the extrinsic curvature) and the Ashtekar formulation, where the set of variables is given by a self-dual connection and a (density weighted) triad. Quantization is accomplished by constructing a Hilbert space of quantum states with an appropriate inner-product such that the phase space variables are promoted to well-defined operators. However, the detailed procedure faces two technical problems. First, general relativity has an infinite number of degrees of freedom. Second, there are non trivial constraints, which are difficult to deal with in the quantum theory. Indeed, the nature of constraints is such that both the procedure of solving the constraints classically prior to quantization and the method of imposing them as operator conditions on states a la Dirac have proven to be difficult to implement.

To gain insight into some of these issues, a number of models have been discussed in the literature. Typically, they are obtained from 3+1-dimensional general relativity

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by symmetry reductions. Perhaps the simplest example is Bianchi models, where spatial homogeneity forces the system to have only a finite number of degrees of freedom. They have proved to be useful to address technical and conceptual questions concerning quantum cosmology [1]. However, since they have only a finite number of degrees of freedom, they are not suitable for investigation of issues concerning the field aspect of gravity. To face this difficulty, it is natural to consider symmetry reductions which are mild enough to leave behind *local* degrees of freedom. Spacetimes with two commuting, space-like Killing vectors fall in this category. In this case, 3+1 dimensional Einstein's equations reduce to a system of equations for a two dimensional field theory. It has been investigated at the classical level [2, 3] and at the quantum level using the Ashtekar formalism in [4].

The next obvious restriction is to assume that the Killing vectors are hypersurface orthogonal. Then the situation simplifies dramatically. For, now, in effect, all the non-linearities go away. One can gauge fix the system in such a way that only one of Einstein's equation has to be solved, which, furthermore, is equivalent to a *non-interacting* scalar field propagating on a *flat* 2+1 dimensional space-time. Since this situation is simple enough to be exactly soluble, it provides a concrete arena to examine the issues of quantum gravity and to see how they can be resolved in practice.

In the spatially non-compact case with $R \times U(1)$ isometry group, such space-times were rigorously quantized in [5] following the pioneer work on the subject by Kuchař[6] in 1971 and later work by Allen [7]. They represent (one polarization) cylindrical gravitational waves, first discovered by Einstein and Rosen. By performing an analysis similar to [5], spacetimes with topology $R^2 \times T^2$ and toroidal symmetries were quantized in [8].

A natural analog of Einstein-Rosen waves but with compact spatial topology is obtained by taking $U(1) \times U(1)$ isometry group acting on a compact manifold $\Sigma = T^3$. Such space-times were initially considered by Gowdy [2] and later canonically quantized by Berger [9]. In the symmetry reduction language used above, the effective linear scalar field now propagates on a flat but expanding space-time. In [9] this field was quantized a la Fock at each instant of time of the flat background metric and the emphasis was on the resulting “particle creation” phenomenon. However, these “particles” do not have a natural physical significance. Indeed, since physically one is dealing here with a closed system—the symmetry reduced gravitational field—one would expect the *physical* vacuum to be stable; there is no physical external field to pump energy into the system and create physical quanta/particles.

In this paper, our purpose is to provide an alternative quantization of the 1-polarization Gowdy models which, we believe, is appropriate for the physics of the problem. In particular, there will be a *single* Hilbert space and no particle creation. We will use the resulting theory to analyze the conceptual and technical problems of quantum gravity. Our analysis is closely related to that of the Einstein-Rosen waves [5] and at appropriate points, we will compare and contrast the technical steps in the

two constructions and conceptual meaning of the final results.

The outline of the paper is as follows. In section 2, we will obtain the reduced phase-space for the system from a 2+1-dimensional perspective. We will see that, before gauge fixing, the two models (Gowdy and Einstein-Rosen) have formally the same Lagrangian. But, the topological differences in the isometry groups and in the spatial slices introduce important technical and conceptual changes. For instance, because of the absence of boundary terms in the action in the spatially compact Gowdy space-times, there is no longer a generator of dynamics on the constraint surface. Therefore, in contrast to the analysis of Einstein-Rosen waves of [5], we will now have to introduce a ‘deparametrization’ procedure to discuss dynamics. Also, unlike Einstein-Rosen waves, the Gowdy space-times have an initial singularity. Finally, even after deparametrization, we will find that a new global constraint and a global degree of freedom is left over, to be carried on to the quantum theory. The model is quantized in section 3. We will see that there is a ‘natural’ Hilbert space for the reduced model. The physical Hilbert space will be the subspace of the Hilbert space corresponding to the kernel of the global constraint. In the quantum theory we will investigate some important issues concerning quantum geometry and coherent states. Finally, in section 4 we will summarize the main results and point out open questions.

2 Hamiltonian Formulation

2.1 The Gowdy system from a 2+1-dimensional perspective

The 3+1-dimensional Gowdy spacetimes considered here have two commuting hypersurface orthogonal space-like Killing fields. This system is equivalent to axis-symmetric solutions of 2+1-dimensional general relativity coupled to a zero rest mass scalar field (which is given by the logarithm of the norm of the Killing field). Let us begin by specifying our midi-superspace from the 2+1 perspective. Thus, we will consider solutions of 2+1-dimensional general relativity with $U(1)$ isometry group coupled to zero rest mass scalar-fields (where the $U(1)$ Killing field is hypersurface orthogonal). The underlying manifold M will be topologically $T^2 \times \mathbb{R}$ and the space-time metric will have signature $(-, +, +)$.

Denote by σ^a the Killing field. Hypersurface orthogonality of σ^a implies that the space-time metric g_{ab} has the form:

$$g_{ab} = h_{ab} + \tau^2 \nabla_a \sigma \nabla_b \sigma \quad (1)$$

where τ is the norm of the Killing vector field and σ is an angular coordinate with range $0 \leq \sigma < 2\pi$ such that $\sigma^a \nabla_a \sigma = 1$. The field h_{ab} so defined is a metric of signature $(-, +)$ on the 2-manifolds orthogonal to σ^a . Let us introduce a generic slicing by compact space-like hypersurfaces labelled by $t = \text{const}$ and a dynamical vector field $t^a = N n^a + N^\theta \hat{\theta}^a$ where n^a is a unit normal to the slices and $\hat{\theta}^a$ is the

unit vector field within each slice orthogonal to σ^a . The pair N, N^θ constitute the lapse and shift. If we now introduce an angular coordinate such that $\hat{\theta}^a \nabla_a \theta = 1$ and $0 \leq \theta < 2\pi$. Then, the 2-metric can be written as:

$$h_{ab} = (-N^2 + N^\theta N_\theta) \nabla_a t \nabla_b t + 2N_\theta \nabla_{(a} t \nabla_{b)} \theta + e^\gamma \nabla_a \theta \nabla_b \theta, \quad (2)$$

where N, N^θ and γ are functions of θ and t . It is because of the underlying $U(1)$ symmetry that the 3-metric g_{ab} has only four independent components and they are functions only of two variables. Moreover due to compactness of the spatial slice they are periodic functions of θ .

Thus, our midi-superspace consists of five functions, $(N, N^\theta, \gamma, \tau, \psi)$ of t and θ , (which are periodic in θ), where ψ is the zero rest mass scalar field. These five fields are subject to the following field equations:

$$G_{ab} = T_{ab}, \quad \text{and} \quad g^{ab} \nabla_a \nabla_b \psi = 0, \quad (3)$$

where G_{ab} is the Einstein tensor of g_{ab} which is determined by the fields (N, N^θ, γ, R) via (1) and (2) and T_{ab} is the stress-energy tensor of the scalar field:

$$T_{ab} = \nabla_a \psi \nabla_b \psi - \frac{1}{2} (g^{cd} \nabla_c \psi \nabla_d \psi) g_{ab}. \quad (4)$$

Note that T_{ab} satisfies the *strong energy condition*, i.e., $T_{ab} \lambda^a \lambda^b \geq -\frac{1}{2} T$, where λ^a is any unit time-like vector field and $T = T^a_a$. Now due to compactness of the spatial slice and the *strong energy condition*, the Hawking-Penrose theorems tell us that the space-time described above will generically have a singularity. This is a major difference from Einstein-Rosen cylindrical waves. There the spatial slices were asymptotically flat and at the classical level we were mainly concerned with an appropriate description of the asymptotic structure. Another difference related to compactness of the spatial slices is that now $\nabla_a \tau$ has to be time-like on M [10] whereas on the non-compact case it is space-like (it is for this reason that at Ref. [5] τ is denoted by R).

Before we conclude this section we would like to comment on the coordinatization of the midi-superspace used here. Let us look at this system from a $3+1$ perspective. In this case the midi-superspace (γ, τ, ψ) refers to vacuum general relativity. There exists in the literature coordinatizations for this model which differ somewhat from ours [2, 9]. But the route to quantization is not as direct as the one that we will obtain here. More details of this comparison will be given later in section 2.4.

2.2 Canonical Form of the Action

Let us begin with the 3-dimensional action:

$$S(g, \psi) := \frac{1}{2\pi} \int_M d^3x \sqrt{g} [\mathcal{R} - g^{ab} \nabla_a \psi \nabla_b \psi], \quad (5)$$

where \mathcal{R} is the scalar curvature of g . There are no boundary terms because the spatial slice is compact. To pass to the Hamiltonian formulation, one performs a 2+1-decomposition. Let us substitute the form of the metric given by Eqs. (1) and (2) in (5). Then, the action reduces to the standard form

$$S = \int dt \left(\int d\theta (p_\gamma \dot{\gamma} + p_\tau \dot{\tau} + p_\psi \dot{\psi}) - H[N, N^\theta] \right), \quad (6)$$

The Hamiltonian H is given by:

$$H[N, N^\theta] = \int d\theta (NC + N^\theta C_\theta) \quad (7)$$

where C and C_θ are functions of the canonical variables:

$$\begin{aligned} C &= e^{-\gamma/2} (2\tau'' - \gamma' \tau' - p_\gamma p_\tau) + \tau e^{-\gamma/2} \left(\frac{p_\psi^2}{4\tau^2} + \psi'^2 \right), \\ C^\theta &= e^{-\gamma} (-2p'_\gamma + \gamma' p_\gamma + \tau' p_\tau) + e^{-\gamma} p_\psi \psi'. \end{aligned} \quad (8)$$

(Here primes denote derivatives with respect to θ .)

As expected the lapse and shift N, N^θ appear as Lagrange multipliers; they are not dynamical variables. Thus, the phase-space Γ consists of three canonically-conjugate pairs of periodic functions of θ , $(\gamma, p_\gamma; \tau, p_\tau; \psi, p_\psi)$ on a 2-manifold Σ which is topologically T^2 . By varying the action with respect to lapse and shift we obtain as usual two first class constraints $C = 0$ and $C^\theta = 0$. (Because of the underlying symmetry of the space-time, the σ component of the diffeomorphism constraint C_σ vanishes identically.) Therefore, the Hamiltonian of the system vanishes on the constraint surface.

Note that, because Σ is compact, now there is no distinction between gauge and dynamics. The situation is different from the one we encountered in the Einstein-Rosen waves. There, the space-time is asymptotically flat and a non-vanishing Hamiltonian generates dynamics, whence the standard gauge fixing procedure is well-suited. In Gowdy space-times, by contrast, Σ is spatially compact and the Hamiltonian vanishes on the constraint surface. Hence one needs to deparametrize the theory, i.e., it is necessary to select a variable or combination of variables on the phase-space to play the role of time. In this sense the compact case is conceptually more subtle and technically more involved.

2.3 Deparametrization

Deparametrization is accomplished by imposing ‘gauge conditions’ on the phase-space variables such that they extract one point from each orbit of the Hamiltonian vector field corresponding to the constraints, *except for one*. This remaining orbit will describe evolution. In another words, from the infinite set of vector fields generated by the Hamiltonian constraints, we have to select one to represent evolution and gauge

fix the others. For gauge fixing, we will choose coordinate conditions to make the space-time geometry transparent. Let us demand:

$$\tau'(\theta) = 0 \quad \text{and} \quad p_\gamma = -p, \quad (9)$$

where p is a spatial constant. The first condition will allow us to regard $\tau(\theta)$ as the time parameter and is motivated by the fact, noted in subsection 2.1, that the field equations imply that $\nabla_a \tau$ is time-like everywhere in M . The second condition states that the scalar density p_γ should equal $-p$, a spatial constant, in our chart (θ, σ) on Σ . It is equivalent to Fourier analyzing p_γ in space and setting all the nonconstant modes in this expansion to zero. It will serve to remove all nonconstant modes of γ from our list of dynamical variables. Let us comment on the presence of this global degree of freedom. The spatial constant p can be expressed as a function on the phase space by integrating the gauge condition on p_γ over the circle, namely $p = -\frac{1}{2\pi} \int_0^{2\pi} p_\gamma d\theta$. Now we can easily verify that p has zero Poisson bracket with all the constraints, i.e., it is a Dirac observable. Therefore it can not be removed by gauge fixing. Thus, if these choices are admissible, apart from a global degree of freedom, the true degrees of freedom will all reside in the field ψ , in accordance with our general expectation that in 2+1 dimensions, all the local degrees of freedom are carried by matter fields. We should add that the conditions (9) were also motivated from the ones adopted previously for Einstein-Rosen waves. A comparison shows that $\tau(\theta)$ is formally replacing the function $R(r)$ in the action. However their roles are quite different due to the restriction on their gradients. Just as we chose $R(r)$ as a radial coordinate, now we choose $\tau(\theta)$ as a time coordinate. Also, in the ER case we set $p = 0$, whereas here, as we will see later on, we are not allowed to do so. This is the source of the global degree of freedom that is not present on the ER case. So, for all the similarities, we might expect to achieve the same technical simplifications on this model.

To see if the coordinate condition is acceptable we have to show that the Poisson brackets $\{\tau'(\theta), H[N, N^\theta]\}$ and $\{p_\gamma + p, H[N, N^\theta]\}$ vanish for a unique choice of lapse and shift, or equivalently $\dot{\tau} = 1$ and $\dot{p}_\gamma = 0$. Explicitly,

$$\begin{aligned} \dot{\tau}(\theta) = \{\tau, H[N, N^\theta]\} &= -N e^{-\gamma/2} p_\gamma + N^\theta e^{-\gamma} \tau' = 1 \\ \dot{p}_\gamma(\theta) = \{p_\gamma, H[N, N^\theta]\} &= \frac{1}{2} N C - \left(N e^{-\gamma/2} \tau' \right)' \\ &\quad + \left(N^\theta p_\gamma \right)' = 0. \end{aligned} \quad (10)$$

As needed the right sides vanish for a unique choice of lapse and shift and do not vanish in any other circumstance. The only solutions are:

$$N = \frac{e^{\gamma/2}}{p} \quad \text{and} \quad N^\theta = 0. \quad (11)$$

Hence the choice (9) selects uniquely a Hamiltonian vector field that represents evolution and fixes the remainder gauge freedom. The Hamiltonian vector field corresponding to the lapse and shift given by (11) generates evolution along the one-parameter family of points labelled by $\tau(\theta) = t$. A remark is in order here. From (10) it is clear that the gauge choice $p_\gamma = 0$ is not appropriate in this case. It would not select a hamiltonian vector field to generate evolution. Therefore from now on we restrict ourselves to the sector in which $p \neq 0$.

Finally, let us extract the true degrees of freedom of the theory. In order to accomplish this, we need to eliminate redundant variables by solving the set of second class constraints (8) and gauge conditions (9). By setting $\tau = t$ and $p_\gamma = -p$ in (8), we can trivially solve for p_τ and for all but one of the infinity degrees of freedom of γ in terms of p , ψ and p_ψ (using the Hamiltonian and the diffeomorphism constraints respectively). The result is:

$$p_\tau = -\frac{t}{p} \left(\frac{p_\psi^2}{4t^2} + \psi'^2 \right), \quad (12)$$

$$\gamma(\theta) = \frac{1}{p} \int_0^\theta d\theta_1 p_\psi \psi' + \gamma(0) \quad (13)$$

Let us explain the global degree of freedom left on γ . If we write $\gamma(\theta) = \frac{q}{2\pi} + \bar{\gamma}(\theta)$, where $q = \int_0^{2\pi} d\theta \gamma(\theta)$, to separate out the spatially variable part $\bar{\gamma}$ of γ , then it is clear that we can solve (13) for $\bar{\gamma}$ and we are left with the global degree of freedom q unsolved. This procedure is equivalent to Fourier analyzing γ in space (as a function of θ) and solving for all modes but the zero mode. Substituting (13) in (11), we can also express the lapse N in terms of q , p , ψ and p_ψ . Thus, as expected, the local degrees of freedom reside just in the matter variables. Indeed, the space-time metric is now completely determined by q , p , ψ and p_ψ :

$$g_{ab} = \frac{e^{q+\bar{\gamma}}}{p^2} (-\nabla_a t \nabla_b t + \nabla_a \theta \nabla_b \theta) + t^2 \nabla_a \sigma \nabla_b \sigma, \quad (14)$$

where, from now on, $\bar{\gamma}$ will only serve as an abbreviation for

$$\bar{\gamma}(\theta) = \frac{1}{p} \int_0^\theta d\theta_1 p_\psi \psi' + \bar{\gamma}(0). \quad (15)$$

One can show that the curvature invariant $R_{abcd}R^{abcd}$ for this space-time metric blows up at $t = 0$ (for $\gamma \neq \text{const}$). Thus, as expected, there is an initial singularity.

Recall that the phase-space variables are periodic functions of θ . Therefore, from (15) we obtain a global constraint, that we will denote by P_θ :

$$P_\theta := \bar{\gamma}(2\pi) - \bar{\gamma}(0) \equiv \int_0^{2\pi} d\theta_1 p_\psi \psi' = 0. \quad (16)$$

Note that this extra global constraint arises because the spatial slices are compact. The presence of this extra constraint will make the quantum theory considerably different from the asymptotically flat case described in [5].

2.4 Reduced Phase Space

The reduced phase space Γ_R can be coordinatized by the pairs $(q, p; \psi(\theta), p_\psi(\theta))$. However, the canonical variables are subject to the global constraint $P_\theta = 0$. Therefore, the physical reduced phase space is non-linear and has the structure of a manifold instead of the usual vector space. Because of this non-linearity it is appropriate to postpone the reduction by the constraint to the quantum theory. Then, it will be imposed as an operator condition on the quantum states.

The (non-degenerate) symplectic structure on the reduced phase space Γ_R is the pull-back of the symplectic structure on Γ . Thus,

$$\{q, p\} = 1 \quad (17)$$

$$\{\psi(\theta_1), p_\psi(\theta_2)\} = \delta(\theta_1, \theta_2) \quad (18)$$

on Γ_R . Next, let us write the reduced action by substituting (9) and (12) in (6),

$$S = \int dt \left[p\dot{q} + \int_0^{2\pi} d\theta \left(p_\psi \dot{\psi} - \frac{t}{p} \left(\frac{p_\psi^2}{4t^2} + \psi'^2 \right) \right) \right], \quad (19)$$

where we have been able to carry out the θ integration on the first term as a consequence of the θ independence of p . The reduced Hamiltonian, that we shall denote by H , is explicitly time dependent. Note that the global constraint is conserved under time evolution because $\{P_\theta, H\} = 0$. By varying the action (19) with respect to ψ and p_ψ we obtain the second-order equation of motion for ψ :

$$-\frac{\partial^2 \psi}{\partial T^2} + \frac{\partial^2 \psi}{\partial \theta^2} - \frac{1}{T} \frac{\partial \psi}{\partial T} = 0, \quad (20)$$

where we defined a new time coordinate T on M via a constant rescaling: $T = t/p$. This is exactly the Klein-Gordon equation for a scalar field propagating on a flat background $\overset{\circ}{g}_{ab}$, given by:

$$\overset{\circ}{g}_{ab} = -\nabla_a T \nabla_b T + \nabla_a \theta \nabla_b \theta + T^2 \nabla_a \sigma \nabla_b \sigma. \quad (21)$$

So we have achieved the same conceptual simplification as in the Einstein-Rosen model, i.e., the decoupling of the system. However it is technically different because now the topology of the fictitious background is not \mathbb{R}^3 , the flat space is not globally Minkowskian. As we will see, it is a wedge of Minkowski space-time identified, such that the boundary of the compact spatial slices is time dependent and starts at a singularity. The topology of the fictitious background and the geometry of the singularity on $\overset{\circ}{g}_{ab}$ is more transparent if we change to a coordinate system where the metric is explicitly flat, i.e.,

$$\overset{\circ}{g}_{ab} = (-\nabla_a x_0 \nabla_b x_0 + \nabla_a \theta \nabla_b \theta) + \nabla_a x_2 \nabla_b x_2. \quad (22)$$

The two sets of coordinates are related by $x_0 = T \cosh(\sigma - \pi)$ and $x_2 = T \sinh(\sigma - \pi)$. Let us concentrate on the plane defined by x_0 and x_2 . Due to the angular range of σ , the x_2 coordinate will be identified, i.e., $x_2(0) = x_2(2\pi)$. At $T = 0$ the length of the orbit of $\partial/\partial\sigma$ goes to zero and thus the orbit reduces to a point and the two-surface (torus) is a circle (on θ). Therefore, the space-time has a singularity at $x_0 = T = 0$. In terms of surfaces $T = \text{const}$, as T increases the spatial slices (tori) are expanding.

Although the fictitious metric has a singularity the initial value problem for the scalar field ψ is well posed $\forall T > 0$. The explicit time-dependence of the Hamiltonian reflects the expanding background and the existence of the singularity. So, differently from the asymptotically flat case, there is a ('mild') back-reaction of the gravitational field.

A remark is in order. In the coordinatization adopted in [9] the reduced system also decouples, but naively it is equivalent to a scalar field, denoted there by $B_-(\theta, t)$ ($\psi = \ln \tau - 2\sqrt{3}B_-$), propagating on a curved background. In the light of this result, the fact that we got a scalar field propagating on a flat background might seem surprising. However, changing the time coordinate t via $\ln 2T = 2t$ one can reinterpret the equation of motion of $B_-(\theta, T)$ to that of a scalar field propagating on the flat background $\overset{\circ}{g}_{ab}$.

Finally, for reasons that will be clear in the next section, let us split the reduced phase space into its global and local degrees of freedom by defining the following direct sum: $\Gamma_R = \overset{\circ}{\Gamma} \oplus \bar{\Gamma}$ where $\overset{\circ}{\Gamma}$ is coordinatized by the canonical pair (q, p) and $\bar{\Gamma}$ by $(\psi(\theta), p_\psi(\theta))$. Moreover let us introduce a covariant description of the phase-space $\bar{\Gamma}$. In this approach the phase-space consists of the real solutions of the Klein-Gordon equation (20), i.e.,

$$\psi(\theta, T) = \sum_{m=-\infty}^{\infty} f_m(\theta, T) A_m + f_m^*(\theta, T) A_m^* \quad (23)$$

where A_m 's are arbitrary constants and

$$\begin{aligned} f_0(\theta, T) &= \frac{1}{2} (\ln T - i) \\ f_m(\theta, T) &= \frac{1}{2} H_0^{(1)}(|m|T) e^{im\theta} \\ &= \frac{1}{2} (J_0(|m|T) + iN_0(|m|T)) e^{im\theta} \quad \text{for } m \neq 0 \end{aligned} \quad (24)$$

where $H_0^{(1)} = J_0 + iN_0$ is the 0th-order Hankel function of the 1st kind and J_0 and N_0 are the 0th-order Bessel function of the first and second kind respectively. (* denotes complex conjugation.).

The symplectic structure is given by:

$$\Omega(\psi_1, \psi_2) = \int_0^{2\pi} T d\theta (\psi_2 \partial_T \psi_1 - \psi_1 \partial_T \psi_2). \quad (25)$$

This covariant approach is completely equivalent to the canonical description outlined on the first paragraph of this subsection. We will refer to the covariant phase space as the ‘space of real solutions’ and denote by V . Note that because the symplectic structure is conserved ($\dot{\Omega} = 0$), generically the scalar field ψ is expected to diverge at $T = 0$.

3 Quantum Theory

3.1 Fiducial Hilbert Space

In this subsection we will ignore the global constraint (16) and quantize the system. The Hilbert space thus obtained will be called fiducial Hilbert space and will be denoted by \mathcal{F} . The physical Hilbert space is the subspace of the fiducial Hilbert space corresponding to the kernel of the global constraint and will be constructed on the next subsection.

Recall that in addition to the local degrees of freedom $(\psi(\theta), p_\psi(\theta))$, which will be treated on the next paragraph as operators acting on a Hilbert space $\bar{\mathcal{F}}$, we also have the global degree of freedom (q, p) , whose quantization is trivial. We will denote by $\mathring{\mathcal{H}}$ the Hilbert space where \hat{q} and \hat{p} are well defined operators. The fiducial Hilbert space, therefore, will be the tensor product of $\mathring{\mathcal{H}}$ and $\bar{\mathcal{F}}$, i.e., $\mathcal{F} = \mathring{\mathcal{H}} \otimes \bar{\mathcal{F}}$.

We will now construct $\bar{\mathcal{F}}$. The overall procedure was discussed in Ref. [5]. Recall that in order to quantize the (unconstrained) theory the phase space variables $(\psi(\theta), p_\psi(\theta))$ are smeared with test fields and represented by operators satisfying the commutation relation corresponding to their Poisson brackets. Quantization is then accomplished by constructing a Hilbert space of quantum states that corresponds to a *-representation of this observable algebra. The appropriate representation will be selected by imposing physical requirements. We will demand the time dependent Hamiltonian (19) and the global constraint (16) to be promoted to well-defined operators.

Recall that the fictitious background (M, \mathring{g}_{ab}) is a suitably identified wedge of flat space-time and that the spatial slices have a time dependent boundary. Moreover there is no global time-like Killing vector field. Therefore, the procedure to obtain a representation, followed in [5], is not directly applicable. We will instead adopt a prescription that is outlined in [11]. We should add that for linear theories both procedures are general and it is just a matter of one being more convenient than the other for the given problem.

The first assumption to obtain a representation for the observable algebra is that, as in [5], the Hilbert space will have the structure of a symmetric Fock-space $\bar{\mathcal{F}}$, based on some one-particle Hilbert space \mathcal{H} . The one-particle Hilbert space can be constructed from the space of real solutions V in the following way. First one introduces on V a complex structure \mathcal{J} which is compatible with the symplectic

structure (25), i.e., $(\psi_1, \psi_2) := \Omega(\mathcal{J}\psi_1, \psi_2)$ is a positive-definite inner product on V . Then, one can define on the complex vector space (V, \mathcal{J}) the inner-product:

$$\langle \psi_1 | \psi_2 \rangle := \frac{1}{2} \Omega(\mathcal{J}\psi_1, \psi_2) + \frac{1}{2} i \Omega(\psi_1, \psi_2). \quad (26)$$

Finally, the one-particle Hilbert space is the Cauchy completion of the complex inner-product space $(V, \mathcal{J}, \langle | \rangle)$. Furthermore, to complete the construction of the Fock space one introduces positive and negative frequency decomposition on \mathcal{H} via:

$$\psi^+ := \frac{1}{2}(\psi - i\mathcal{J}\psi) \quad (27)$$

$$\psi^- := \frac{1}{2}(\psi + i\mathcal{J}\psi) \quad (28)$$

such that $\psi = \psi^+ + \psi^-$. Thus, a field operator in the Fock space is represented by $\hat{\psi}(\psi) = \hat{A} + \hat{C}$, where \hat{C} and \hat{A} are creation and annihilation operators associated with the positive and negative frequency decomposition.

In summary, to obtain a representation all the work can be focussed on finding an appropriate complex structure on V . In our model there is a natural complex structure that arises from the general solution (23). Using the fact that the solution in (23) naturally occurs in pairs, we define a complex structure \mathcal{J} for our model as:

$$\mathcal{J} \alpha \ln T := -\alpha \quad \text{and} \quad \mathcal{J} \alpha := \alpha \ln T \quad \text{for } m = 0 \quad (29)$$

$$\mathcal{J} J_0(|m|T) := N_0(|m|T) \quad \text{and} \quad \mathcal{J} N_0(|m|T) := -J_0(|m|T) \quad \text{for } m \neq 0 \quad (30)$$

(α is a constant.) Note that $J^2 = -1$ as required. Thus, the “positive frequency” part can be obtained by using (23), (27), (29) and (30):

$$\psi^+ = \sum_{m=-\infty}^{\infty} f_m^*(\theta, T) A_m^*, \quad (31)$$

and, from (28), the negative frequency part is just the complex conjugate of (31). Moreover one can show that the complex-structure, given by (29, 30), is in fact compatible with the symplectic structure and therefore completely determines the inner-product on the one particle Hilbert space \mathcal{H} . Following the prescription we can write the field operator $\hat{\psi}$ in terms of creation and annihilation operators corresponding to the positive and negative frequency decomposition defined on \mathcal{H} by the complex structure (29, 30):

$$\hat{\psi}(\theta, T) = \sum_{m=-\infty}^{\infty} \left(f_m(\theta, T) \hat{A}_m + f_m^*(\theta, T) \hat{A}_m^\dagger \right) \quad (32)$$

A comparison with the classical solution (23) shows that one could obtain this field operator naively by promoting the constants of motion from the explicit solution to

operators. In fact this is also a motivation for choosing (29, 30) as the complex structure. Furthermore, as needed, one can show that the normal ordered Hamiltonian and global constraint are well-defined operators.

There is an important result from the above construction. Note that the complex-structure is time *independent*, therefore, there is no mixing between positive and negative parts or analogously, creation and annihilation operators as time passes. Thus, in contrast to ref.[9], we conclude that there is no-creation of ‘particles’.

3.2 Physical Hilbert Space

The space \mathcal{F}_{phys} of physical states is the subspace of the fiducial Hilbert space \mathcal{F} defined by:

$$: \hat{P}_\theta : |\Psi\rangle_{phys} = 0. \quad (33)$$

In order to have a better understanding of this condition let us express the global constraint in terms of creation and annihilation operators:

$$: \hat{P}_\theta : |\Psi\rangle_{phys} = 2 \sum_{m=-\infty}^{\infty} m \hat{A}_m^\dagger \hat{A}_m |\Psi\rangle_{phys} = 0. \quad (34)$$

Thus, the physical states are states such that the total angular momentum in θ -direction vanishes. Therefore, obviously the (usual) vacuum state defined by $\hat{A}_n \otimes \hat{q}|0\rangle = 0$, $\forall k$ and states with particles in the zero mode (i.e. $m = 0$) belong to the physical Hilbert space. Explicitly, a generic physical state with N particles is given by:

$$|^N \Psi\rangle_{phys} = |\phi\rangle \otimes \left[\prod_{i=1}^N |m_i\rangle \right] \quad \text{such that} \quad \sum_{i=1}^N m_i = 0. \quad (35)$$

where $|\phi\rangle$ is a state that belongs to \mathcal{H} and $|m_i\rangle$ represents a one-particle state with angular momentum m_i . Note that, the space of physical states does not inherit the Fock space structure of $\bar{\mathcal{F}}$. For instance, except for the zero mode, none of the one-particle states of $\bar{\mathcal{F}}$ belongs to \mathcal{F}_{phys} . Nonetheless because the orbit of P_θ is compact (or $: \hat{P}_\theta :$ has discrete spectrum) \mathcal{F}_{phys} is a closed subspace of \mathcal{F} and hence has a natural Hilbert space structure. Moreover, the operators from \mathcal{F} can be projected to \mathcal{F}_{phys} . We will denote the projection operator by \mathcal{P} .

Let us now investigate the dynamics of the system. The Hamiltonian, from expression (19), can be promoted to the operator:

$$\hat{H}(T) = \int_0^{2\pi} d\theta : \hat{p} T \left(\frac{\hat{p}_\psi^2}{4\hat{p}^2 T^2} + (\hat{\psi}')^2 \right) :, \quad (36)$$

where we have rescaled the time as before. Expressing in terms of creation and annihilation operators, we obtain:

$$\begin{aligned}\hat{H} &= \frac{\pi}{2T} : \hat{p}(\hat{A}_0 + \hat{A}_0^\dagger)^2 : \\ &+ \frac{T}{4} \sum_{m=-\infty}^{\infty} m^2 \hat{p} \left[2\hat{A}_m^\dagger \hat{A}_m (H_1^{(1)} H_1^{(2)} + H_0^{(1)} H_0^{(2)}) \right. \\ &\left. + \hat{A}_m \hat{A}_{-m} \left((H_0^{(1)})^2 + (H_1^{(1)})^2 \right) + \hat{A}_m^\dagger \hat{A}_{-m}^\dagger \left((H_0^{(2)})^2 + (H_1^{(2)})^2 \right) \right], \quad (37)\end{aligned}$$

where $H_0^{(2)}(|m|T) = [H_0^{(1)}(|m|T)]^*$ and $H_1^{(1,2)}(|m|T) = -\frac{1}{|m|} \dot{H}_0^{(1,2)}(|m|T)$. By inspection one can easily conclude that the action of the Hamiltonian operator on a physical state gives back a physical state, thus it leaves \mathcal{F}_{phys} invariant. Note that the vacuum state is *not* an eigenvector of the Hamiltonian with zero eigenvalue. However, this Hamiltonian arises from our choice of deparametrization, therefore apriori there is no direct physical significance to this result. In contrast to the Einstein-Rosen model, where the vacuum was physically defined, here it is not clear how to define the ‘real’, physical vacuum. This is a consequence of the arbitrariness of the deparametrization procedure.

In order to investigate the time parameter that arises from the quantum theory let us write the Schrödinger equation associated with (36):

$$i\hbar \frac{\partial}{\partial T} |\Psi\rangle_{phys}(T) = \hat{H}(T) |\Psi\rangle_{phys}(T). \quad (38)$$

One can interpret this equation by saying that this evolution takes place on the fictitious background $\overset{\circ}{g}_{ab}$ where $\frac{\partial}{\partial T}$ has a space-time interpretation. The physical space-time is a derived quantity, i.e., there is no physical metric to start with, thus apriori $\frac{\partial}{\partial T}$ has no physical interpretation. Nonetheless, because of the classical decoupling we can still write the metric operator by using the chart (T, θ, σ) . This is remarkable because we can conclude that in fact, although there is no background physical metric, the simplification of this system is such that provides a time parameter for the quantum theory.

3.3 Quantum Geometry

Let us start investigating semi-classical geometries. In order to do this, we shall return to the fiducial Hilbert space \mathcal{F} . First we promote the classical expression of the space-time metric (14) to an operator on \mathcal{F} . Formally,

$$“\hat{g}_{ab} =: e^{\hat{q} + \hat{\gamma}(\theta, T)} : (-\nabla_a T \nabla_b T + \nabla_a \theta \nabla_b \theta) + \hat{p}^2 T^2 \nabla_a \sigma \nabla_b \sigma” \quad (39)$$

As in the Einstein-Rosen case, the states in $\bar{\mathcal{F}}$ yielding semi-classical geometry will be the usual coherent states for the field operators. Similarly, the matrix elements of this operator on coherent states on $\bar{\mathcal{F}}$ are well-defined. In particular, the expectation

value on a coherent state $|\Psi_c\rangle$ in \mathcal{F} , gives the classical expression (14) (assuming the time rescaling) evaluated on the classical field configuration ψ_c and the mean values $\langle \hat{q} \rangle$ and $\langle \hat{p}^2 \rangle$ peaked on a classical solution q and p^2 , explicitly

$$\langle \Psi_c | \hat{g}_{ab} | \Psi_c \rangle = e^{q+2 \int_0^\theta T \dot{\psi}_c \psi'_c d\theta_1 + \bar{\gamma}(0)} (-\nabla_a T \nabla_b T + \nabla_a \theta \nabla_b \theta) + p^2 T^2 \nabla_a \sigma \nabla_b \sigma. \quad (40)$$

An immediate consequence of this result is that the coherent state provides us an example to show that the singularity persists in the quantum theory. Specifically, we can compute a scalar formed out from the Riemann tensor, promote to a (normal ordered) quantum operator and calculate its expectation value on a coherent state. We will, then, obtain the corresponding divergent classical value.

The only non-trivial metric operator component is $e^{\hat{q}} \exp[(\int_0^\theta 2T \dot{\hat{\psi}} \hat{\psi}' d\theta_1) + \hat{\gamma}(0)]$. Note that the exponent of the second factor has the same functional form as the angular momentum for a scalar field in a box (if one recovers the integral on σ that is omitted due to symmetry). Similarly to the energy in a box that was extensively discussed in Ref. [5] and [12], one can show that it is not a well-defined operator on $\bar{\mathcal{F}}$. Thus, after regularizing we obtain the regulated metric operator on $\bar{\mathcal{F}}$:

$$\hat{g}_{ab}(f_\theta) = e^{\hat{q}} e^{\int_0^{2\pi} f_\theta(\theta_1) 2T : \dot{\hat{\psi}} \hat{\psi}' : d\theta_1 + \hat{\gamma}(0)} (-\nabla_a T \nabla_b T + \nabla_a \theta \nabla_b \theta) + \hat{p}^2 T^2 \nabla_a \sigma \nabla_b \sigma, \quad (41)$$

where the regulator $f_\theta(\theta_1)$ equals 1 for $\theta_1 \leq \theta - \epsilon$, then it smoothly decreases to zero and equals zero for $\theta_1 \geq \theta - \epsilon$. Also, $f_{2\pi}(\theta) = f_0(\theta)$. As we see, again the Planck length comes into play naturally in the quantum theory

Although we have obtained a well-defined operator on $\bar{\mathcal{F}}$, it is not an operator on the physical space because it does not commute with the global constraint $:\hat{P}_\theta:$. However, as we pointed out before, it can be projected to the physical space. Therefore the physical regulated metric operator in \mathcal{F}_{phys} is given by:

$$[\hat{g}_{ab}(f_\theta)]_{phys} = \mathcal{P} \hat{g}_{ab}(f_\theta) \mathcal{P}. \quad (42)$$

As in the Einstein-Rosen waves, there are interesting features associated to the metric operator. Because the calculations are similar, we will avoid the repetition by pointing out only the final results. First there will be quantum fluctuations of the light-cone. Specifically, for a given quantum state of the system (gravity coupled to scalar field), the norm of a null vector will fluctuate between positive, null and negative values. Second the commutator between two non-trivial metric operators is non-vanishing, this is a consequence, as before, of the non locality of the metric components with respect to the (basic) scalar field. Finally, the holonomy operator is well-defined in this model as well. Regarding this operator there are some differences that are worth pointing out in more detail. Recall that in the Einstein-Rosen waves the first internal gauge choice that we made led to a badly-behaved connection at the origin (Eq. (24) in Ref. [5]). Therefore, a gauge transformation was necessary. After

that, the component of the connection along $\nabla_a R$ was not Abelian anymore, therefore the holonomy along a loop $\sigma = \text{const}$ was not trivial to compute. It was given by the path-ordered exponential. Thus, we explicitly computed only the holonomy along a loop defined by $R = \text{const}$ because in this case the component of the connection that was contributing was Abelian. However, now using the same (first) internal gauge choice, we obtain a well-defined connection because of the topology of the spatial slices. Moreover, the holonomy along both generators of the torus can be easily computed, because the respective contributing components of the connection are Abelian. The corresponding operators are given by:

$$\hat{T}_\eta^0 = 2 \cosh \left[\pi e^{-i\hat{\gamma}(f_\theta)/2} \right] :, \quad (43)$$

for a loop η with tangent vector given by $\dot{\eta}^a = \sigma^a$ (along the integral curve of the Killing field), and

$$\hat{T}_\eta^0 = 2 \cosh \left[\frac{\hat{H}}{\hat{p}} \right], \quad (44)$$

where now the loop η has tangent vector given by $\dot{\eta}^a = \theta^a$ and \hat{H} is the Hamiltonian operator (36). Note that in the Einstein-Rosen case we calculated the holonomy for a loop along the integral curve of the Killing field and it needed to be regulated. Here, the operator (43) has to be regulated as well to yield a well-defined operator on $\bar{\mathcal{F}}$. Moreover it does not commute with the global constraint, therefore it has to be projected to \mathcal{F}_{phys} . On the other hand, the holonomy operator (44) does not require any regularization procedure (other than normal ordering). Moreover, it commutes with the global constraint. Therefore it is automatically a well-defined operator on \mathcal{F}_{phys} , as expected (see discussion in Ref. [5]).

3.4 Physical Coherent States

In this section we will obtain the physical coherent states. In the fiducial Hilbert space, the coherent states were exactly the usual coherent states for the field operator in $\bar{\mathcal{F}}$ tensor product with a quasi-classical state in $\mathring{\mathcal{H}}$. However, in general, the field operator $\hat{\psi}(\theta, T)$, or equivalently the annihilation and creation operators, \hat{A}_m and \hat{A}_m^\dagger (for $m \neq 0$), do not leave the physical Hilbert space invariant. Therefore, to obtain a coherent state on \mathcal{F}_{phys} we will adopt the following strategy: First we will obtain a set of ‘basic’ operators such that any operator on $\bar{\mathcal{F}}_{phys}$ can be expressed as a combination of them. They play a role on $\bar{\mathcal{F}}_{phys}$ analogous to \hat{A}_m on $\bar{\mathcal{F}}$. Then, having obtained these operators, we will seek states where they will be peaked on their classical value. These will be the coherent state on $\bar{\mathcal{F}}_{phys}$ because by construction all other normal ordered operators will be peaked on their classical value as well. Finally we will obtain the coherent state on $\mathcal{F}_{phys} = \mathring{\mathcal{H}} \otimes \bar{\mathcal{F}}_{phys}$ in the obvious way.

Let us obtain the set of ‘basic’ operators on $\bar{\mathcal{F}}_{phys}$. Note, first, that classically the phase space $\bar{\Gamma}$ modulo the global constraint P_θ can be coordinatized by the infinite

set:

$$\beta_{\{m_i\}}^{[N]} = \prod_{i=1}^N A_{m_i} \quad \text{and} \quad [\beta_{\{m_i\}}^{[N]}]^* = \prod_{i=1}^N A_{m_i}^* \quad \text{with} \quad \sum_{i=1}^N m_i = 0 \quad (45)$$

(The infinite set A_m, A_m^* is subject to *one* global constraint, thus we are still left with an infinite set.) Now, the ‘basic’ set of operators can be obtained by promoting (45) to operators on $\bar{\mathcal{F}}_{phys}$:

$$\begin{aligned} \hat{\beta}_{\{m_i\}}^{[N]} &= \prod_{i=1}^N \hat{A}_{m_i} \quad \text{with} \quad \sum_{i=1}^N m_i = 0 \\ [\hat{\beta}_{\{m_i\}}^{[N]}]^\dagger &= \prod_{i=1}^N \hat{A}_{m_i}^\dagger \quad \text{with} \quad \sum_{i=1}^N m_i = 0 \end{aligned} \quad (46)$$

We can now ask if there exists a state on $\bar{\mathcal{F}}_{phys}$ with the following property,

$$\hat{\beta}_{\{m_i\}}^{[N]} |\beta_c\rangle = \beta_{\{m_i\}}^{[N]} |\beta_c\rangle. \quad (47)$$

This will be the coherent state that we are looking for. The answer is that there exists such state and it is given explicitly by:

$$\begin{aligned} |\beta_c\rangle &= |0\rangle + \beta_0^{[1]} |1(0)\rangle + \sum_{m=-\infty}^{\infty} \beta_{\{m\}}^{[2]} |1(m), 1(-m)\rangle + \dots \\ &+ \dots + \sum_{\{J\}} \frac{\beta_{\{J\}}^{[N]}}{\sqrt{N_1!} \dots \sqrt{N_I!}} |N_1(1), \dots, N_I(I), \dots\rangle + \dots \end{aligned} \quad (48)$$

where the sum stands for all possible sets $\{J\}$ such that

$$\sum_{J=-\infty}^{\infty} J N_J = 0 \quad \text{and} \quad \sum_{J=-\infty}^{\infty} N_J = N. \quad (49)$$

We have changed the notation slightly in order to adopt the basis of the number operator. To make the notation clear, let us give an example of two possible sets of this sum with three particles, i.e., $N = 3$:

$$\frac{1}{\sqrt{2}} \beta_{(1,1,-2)}^{[3]} |2(1), 1(-2)\rangle \quad \text{and} \quad \beta_{(12,-4,-8)}^{[3]} |1(12), 1(-4), 1(-8)\rangle. \quad (50)$$

Finally, a coherent state $|\Psi_c\rangle_{phys}$ on \mathcal{F}_{phys} will be given by $|\Psi_c\rangle_{phys} = |\beta_c\rangle \otimes |\phi_c\rangle$, where $|\phi_c\rangle$ belongs to \mathcal{H} and is such that as before \hat{q} and \hat{p} are peaked on a classical solution given by q and p .

The expectation value of the physical metric operator given by (41) on $|\beta_c\rangle$ yields by construction:

$${}_{phys} \langle \Psi_c | [\hat{g}_{ab}]_{phys} | \Psi_c \rangle_{phys} = [g_{ab}(q, p, \beta_c, \beta_c^*)]_{phys}, \quad (51)$$

i.e., classical geometries on the phase space with the global constraint implemented. Note that, as before, there is no need to have the regulator in this case.

Another interesting result is that one can show that the physical coherent state is the projection to \mathcal{F}_{phys} of the coherent state ($\langle \Psi_c | \hat{\psi} | \Psi_c \rangle = \psi_c$) on $\tilde{\mathcal{F}}$, i.e.,

$$|\Psi_c \rangle_{phys} = \mathcal{P} |\Psi_c \rangle. \quad (52)$$

In particular, note that if ψ_c is a classical solution corresponding to the zero mode, that we will denote by ${}^0\psi_c$, then $|\Psi_c \rangle_{phys} = |{}^0\Psi_c \rangle$, i.e., the physical coherent state for this classical solution is equivalent to the (usual) coherent state.

As a consequence of (52) the expectation value of any operator projected to \mathcal{F}_{phys} on a physical coherent state has the following correspondence with respect to the operator and coherent states on \mathcal{F} :

$${}_{phys} \langle \Psi_c | \hat{O}_{phys} | \Psi_c \rangle_{phys} = \langle \Psi_c | \hat{O}_{phys} | \Psi_c \rangle = {}_{phys} \langle \Psi_c | \hat{O} | \Psi_c \rangle_{phys}. \quad (53)$$

4 Discussion

Our choice of midi-superspace variables was motivated by our previous work on Einstein-Rosen waves [5] and differs from the literature on cosmological models. Our intention was to obtain the same type of simplifications as in the Einstein-Rosen case and this was indeed achieved. We saw that although this model had formally the same bulk Lagrangian density as the Einstein-Rosen waves (with τ replaced by R), because the spatial slices are compact the action is different; it does not have a boundary term. In the absence of a non-vanishing Hamiltonian we need to deparametrize the theory. We chose $\tau(\theta)$ as the time coordinate. In the Einstein-Rosen case $R(r)$ was chosen as the radial coordinate. This difference has a counterpart in space-time language: In the Gowdy models, due to spatial compactness, the gradient of $\tau(\theta)$ is time-like everywhere on M whereas in the Einstein-Rosen waves $\nabla_a R(r)$ is space-like. As is usual in the deparametrization approach, the momentum canonically conjugate to $\tau(\theta)$ turned out to be the Hamiltonian of the system. In the Einstein-Rosen waves, by contrast, $p_R(r)$ does not play a special role.

The reduced system was again remarkably simple. However, there are some fundamental differences from the non-compact case. The infinite number of true degrees of freedom are represented again by the scalar field but now there is in addition one global degree of freedom. The reduced Hamiltonian is explicitly time-dependent. There exists a new global constraint that requires the total angular momentum in θ -direction to vanish. We decided to carry it over as an operator condition on the quantum theory. Next, the space-time now has an initial singularity. Finally, although the decoupling occurs as in the Einstein-Rosen case, now the scalar field propagates on a wedge of flat space-time with certain identifications, rather than on a full Minkowski space-time. An analysis of this identified background space-time

showed that the spatial slices are tori with time-expanding boundary that start at a singular circle.

Because of the topological differences of the fictitious flat background and the absence of a time-like Killing vector field, the procedure to obtain the quantum representation is different. Now, the representation is directly related to the choice of a complex structure. This approach is more suitable for linear theories and does not rely on the staticity of the background whereas the approach used in [5] (to find a suitable measure on the quantum configuration space) is more convenient if there is a static Killing vector field (but can be extended also to non-linear theories). The complex structure for our model was chosen accordingly to physical requirements. This led us to a fiducial Hilbert space. An interesting problem is to work out the corresponding measure in the space of quantum states $L^2(S', d\mu)$.

In contrast to ref. [9], we obtained one fixed Hilbert space, rather than a one-parameter family of them. This is a consequence of the fact that our complex structure is time *independent*. Another implication is that there is no creation of ‘particles’. Indeed, one could obtain particle creation even in Minkowski space-time just by a choice of time dependent complex structure. Here we do not have external fields, i.e., we are not quantizing fields in a curved space-time. Therefore, there is no reason to expect creation of particles or even to interpret the quanta of the scalar field as physical particles.

The physical Hilbert space is the subspace of the fiducial state space defined by the kernel of the global constraint. There are no one-particle physical states, except for the zero mode. We saw that because the orbits of the classical constraint vector field are closed, the quantum physical operators can be obtained by simple projection into the physical subspace. We also obtained semi-classical states. The expectation value of the metric operator in such a state corresponds to classical geometry.

The issue of time in this model is technically simpler but conceptually more subtle. As in the Einstein-Rosen case the gauge choice is such that the problem decouples and a time parameter arises naturally in the quantum theory. It is the time parameter of the fictitious background. The conceptual problem is that while we had a distinction between gauge and dynamics and a notion of time at spatial infinity before, now the time parameter and the Hamiltonian are artifacts of our deparametrization, i.e., they are not singled out by any physical reason. But, as is well known, this problem is intrinsic to the theory itself and is also present in the classical theory. The key point is that we were able to find a *gauge* which selects a fictitious background and provides a global time parameter for the quantum theory as well.

In this model, as we pointed out, there is an initial singularity in all classical solutions, including the flat fictitious background. Nonetheless we were able to quantize the system consistently. Note, however, that the quantization did not cure this break down of the classical theory. However, since the symmetry reductions freeze out degrees of freedom, and hence it is not clear that the problem will persist in the full quantum theory.

A supportive result for the non-perturbative program based on Ashtekar's self-dual connections is that the operator corresponding to the trace of the holonomy of a loop around the axis of symmetry is well-defined on the Hilbert space of both models. The holonomy along a loop in the perpendicular direction, due to axi-symmetry, corresponds to a 2-dimensional smearing. Therefore, it is expected to be well-defined. We were able to compute this holonomy as a simple exponential, and we verified that, in fact, the operator corresponding to its trace is well-defined without need of regularization (other than the usual normal ordering). It is not clear, however, how the first result will change if we adopt a different gauge choice and it may well be that a smearing of the loop would be necessary then. Here we saw that there is no need of such smearing because the metric is a surface integral of the basic fields.

To summarize, this model can be successfully used to probe a number of features of the quantum theory. The key reason is that it is a simple, exactly soluble model. It makes potential problems transparent and suggest methods to deal with them.

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